EPFL & SFI

Asset Pricing Theory

Asset Pricing. Homework 5 Solution.

1. CARA-Normal utility and the CAPM

Assume a standard one-period setup with n risky securities with corresponding $(n, 1)$ vector of prices P (at date 0) and $(n, 1)$ vector of payoffs X (paid at date 1) which are jointly normally distributed with mean μ and a diagonal covariance matrix Σ (only diagonal elements are different from zero $\Sigma_{ii} = \sigma_i^2$). Suppose also there is a risk-free security that pays 1 a.s. at date one and has a price $1/R_f$ (at date 0). Assume there exists K agents who maximize their negative exponential utility $\mathbb{E}[-\exp(-\gamma_k c_k)]$ $k = 1, \dots K$ given some initial endowment of wealth W_k.

Each security i is in a supply of s_i.

1. Derive the optimal consumption and portfolio holdings of the investors.

   We have

$$W_k \;=\; (W_k - \sum_i \pi_i(k)P_i) \;+\; \sum_i \pi_i(k)P_i$$

   whereas

$$c_k \;=\; R_f(W_k - \sum_i \pi_i(k)P_i) \;+\; \sum_i \pi_i(k)X_i \;=\; R_f W_k + \pi_i^\top (X - R_f P)$$

   and the expected utility is

$$E[-e^{-\gamma_k c_k}] \;=\; -e^{-\gamma_k R_f W_k} E[e^{-\gamma_k \pi'(X - R_f P)}]$$

   whereas

$$E[e^{-\gamma_k \pi'(X - R_f P)}] \;=\; e^{-\gamma_k \pi'(\mu - R_f P) + 0.5\gamma_k^2 \pi' \Sigma \pi}$$

   Thus, the optimal portfolio solves

$$\max_\pi \{\pi'(\mu - R_f P) - 0.5\gamma_k \pi' \Sigma \pi\}$$

   and hence

$$\pi \;=\; \gamma_k^{-1} \Sigma^{-1} (\mu - R_f P)$$

2. Derive the equilibrium price vector P assuming that there is one unit of each of the risky securities outstanding.

   Market clearing implies

$$\sum_k \gamma_k^{-1} \Sigma^{-1} (\mu - R_f P) \;=\; s$$

   where $s = (s_i)_{i=1}^n$ is the vector of supply. Hence,

$$\sum_k \gamma_k^{-1} (\mu - R_f P) \;=\; \Sigma s$$

   and

$$R_f P \;=\; \mu \;-\; \bar{\gamma} \Sigma s \;=\; \mu - \gamma \mathbb{C}\mathrm{ov}(X, s'X) \tag{1}$$

where $X_M = s'X$ is the payoff of the **market portfolio** *and*

$$\bar{\gamma} \;=\; (\sum_k \gamma_k^{-1})^{-1}\,.$$

Now, returns are given by $R = X/P \;=\; diag(P)^{-1}X$ whereas the covariance matrix of returns is $\Sigma_R = diag(P)^{-1}\Sigma diag(P)^{-1}$. At the same time, the price and return of the market are

$$P_M \;=\; s'P,\;\; R_M \;=\; X_M/P_M$$

Our goal is to show that

$$E[R_i - R_f] \;=\; \beta_i E[R_M - R_f]$$

or, equivalently,

$$E[R - R_f] \;=\; E[R_M - R_f]\frac{\mathbb{C}\mathrm{ov}(R, R_M)}{\mathbb{V}\mathrm{ar}[R_M]}$$

We have

$$\mathbb{C}\mathrm{ov}(R_i, R_M) \;=\; \mathbb{C}\mathrm{ov}(P_i^{-1}X_i, P_M^{-1}X_M) \;=\; P_i^{-1}P_M^{-1}\mathbb{C}\mathrm{ov}(X_i, X_M)$$

and therefore, in vector form,

$$\mathbb{C}\mathrm{ov}(R, R_M) \;=\; P_M^{-1}\mathrm{diag(P)}^{-1}\Sigma\mathrm{s}$$

And hence the desired identity takes the form

$$E[R - R_f] \;=\; E[R_M - R_f]\frac{\mathbb{C}\mathrm{ov}(R, R_M)}{\mathbb{V}\mathrm{ar}[R_M]} \;=\; \kappa\mathrm{diag(P)}^{-1}\Sigma\mathrm{s}$$

where we have defined

$$\kappa \;=\; \frac{E[R_M - R_f]}{\mathbb{V}\mathrm{ar}[R_M]}P_M^{-1}\,.$$

Substituting

$$E[R - R_f] \;=\; \mathrm{diag(P)}^{-1}\mathrm{E[X - R_f P]} \;=\; \mathrm{diag(P)}^{-1}(\mu - \mathrm{R_f P}),$$

we get

$$\mathrm{diag(P)}^{-1}(\mu - \mathrm{R_f P}) \;=\; \kappa\mathrm{diag(P)}^{-1}\Sigma\mathrm{s}\,,$$

which is equivalent to

$$\mu - R_f P \;=\; \kappa\Sigma s$$

when we multiply by $\mathrm{diag(P)}$*. Thus, to complete the proof, it remains to show that $\kappa = \bar{\gamma}$. We have*

$$\frac{E[R_M - R_f]}{\mathbb{V}\mathrm{ar}[R_M]} \;=\; \frac{P_M^{-1}E[X_M - R_f P_M]}{P_M^{-1}\mathbb{V}\mathrm{ar}[X_M]} = P_M\frac{E[X_M - R_f P_M]}{\mathbb{V}\mathrm{ar}[X_M]}$$

Furthermore, taking at inner product of (1) *with s, we get*

$$R_f s'P \;=\; s'\mu \;-\; s'\bar{\gamma}\Sigma s \tag{2}$$

We notice that

$$s'\Sigma s \;=\; \mathbb{V}\mathrm{ar}[s'X] \;=\; \mathbb{V}\mathrm{ar}[X_M]\,,$$

and hence (2) *is equivalent to*

$$\bar{\gamma}\mathbb{V}\mathrm{ar}[X_M] \;=\; s'\mu - R_f s'P \;=\; E[X_M] - R_f$$

and hence, we get the desired expression for κ.

3. Show that the CAPM holds in this economy, i.e., that if we define $R_i = X_i/P_i \;\; i = 1, \dots n$ we have

$$\mathbb{E}[R_i] = R_f + \beta_i(\mathbb{E}[R_M] - R_f)$$

   where R_M is the return on the market portfolio, which invests a fraction $\omega_i = \frac{P_i}{\sum_{i=1}^n P_i}$ in each security (i.e., in relation to its relative market capitalization).

4. Identify the market risk-premium $\mathbb{E}[R_M] - R_f$ and Sharpe ratio $\frac{E[R_M]-R_f}{\sigma_M}$ in terms of the primitives of the model (and in particular in terms of R_f), $\bar{\gamma} = (\sum_k \gamma_k^{-1})^{-1}$, $\hat{\mu} = \sum_i \mu_i$, and $\hat{\sigma}^2 = \sum_i \sigma_i^2$.

5. Show that there exists a pricing kernel that is linear in the sum of the aggregate dividend, i.e., $M = a - b \sum_{i=1}^n X_i$ where you have to determine a, b, such that $E[M\, R_i] = 1 \, \forall i$.

6. Note that this state price density can take on negative values. Does this imply that there are arbitrage opportunities in this economy? Explain.

7. Are markets complete?

8. Is the equilibrium allocation Pareto optimal?

9. what happens when each agent k has his own view on (μ_k, Σ_k)?

2. The CAPM: Empirical evidence

Read Fama-French (JEP 2004): The Capital Asset Pricing Model: Theory and Evidence (posted on Moodle).

1. Explain figure 2 page 33 in FF2004. What does it represent? What does this figure imply for the CAPM?

2. Explain figure 3 page 43 in FF2004. What does it represent? What does this figure imply for the mean-variance efficiency of the market portfolio?

3. Mean-variance portfolio choice and leverage constraints.

Consider an economy with $N = 2$ risky assets R_1, R_2 and one risk-free asset R_0. The expected return vector is $\mu = [0.09; 0.12]$ and standard deviation $\sigma = [0.15; 0.25]$. The correlation between the two returns is 0.2. There is a risk-free rate $R_0 = 0.05$. We want to solve the problem of a mean-variance investor who faces leverage constraints and cannot borrow more than 20% of her wealth. The investor seeks the portfolio R_P such that $\max E[R_P] - \frac{a}{2} V[R_p]$ subject to $w'\mathbf{1} \leq m$ where $m = 1.2$ and w is the vector of weights invested in the risky assets.

1. Solve the optimal portfolio problem of an unconstrained investor (with $m = \infty$). Show that her optimal portfolio consists of investing in the so-called 'tangency' portfolio (the mean-variance optimal portfolio that holds only risky assets) and the risk-free asset. Determine the tangency portfolio w_t, its mean, variance, and Sharpe ratio. Show that the relative weight put on the tangency portfolio versus the risk-free asset depends on the risk aversion of the investor.
2. Determine the zero beta portfolio w_z, which is mean-variance efficient and has zero correlation with the tangency portfolio. Compute its mean, variance, and Sharpe ratio.
3. Show that for an investor with a leverage constraint, her optimal portfolio consists of a combination of the risky-asset-only tangency portfolio, the zero-beta portfolio, and the risk-free rate. This implies that we can restrict the optimal portfolio choice problem to portfolios with returns of the form $R_P = R_0 + x_t(R_t - R_0) + x_z(R_z - R_0)$. Setup the Lagrangian of the constrained agent's problem, derive the first-order condition and compute the optimal portfolio in terms of x_t, x_z, the holdings of tangency and zero-beta portfolio. Then, also give the portfolio composition in terms of the underlying securities w_0, w_1, w_2.
4. Prove that there exists a risk-aversion level a^* so that if $a > a^*$, then the agent is unconstrained and does not hold the zero-beta portfolio. Instead, if $a < a^*$ then the agent will also invest in the zero-beta portfolio.
5. Plot the Sharpe ratio on the optimal portfolio as a function of the risk-aversion level. What happens to the Sharpe ratio of the optimal portfolio as a falls below a^*? Interpret the findings.

We have

$$w_0 \;=\; 1 - w_1 - w_2 \tag{3}$$

(the weight in the risk-free asset) and the agent is maximizing

$$\max w'\hat{\mu} \;-\; 0.5aw'\Sigma w,\;\; \hat{\mu} = \mu - R_0, \tag{4}$$

where

$$\Sigma \;=\; \mathrm{diag}(\sigma)\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\mathrm{diag}(\sigma) \tag{5}$$

The solution is

$$w_{Markowitz} \;=\; a^{-1}\Sigma^{-1}\hat{\mu} \tag{6}$$

when the agent is unconstrained.

A tangency portfolio can be defined in two ways. One way is to fix a risk aversion and solve

$$\max w'\mu \;-\; 0.5aw'\Sigma w,\;\; w'1 = 1 \tag{7}$$

which gives first-order conditions

$$\hat{\mu} - a\Sigma w - \lambda 1 \;=\; 0 \tag{8}$$

so that the tangency portfolio is

$$w_t \;=\; a^{-1}\Sigma^{-1}(\hat{\mu}-\lambda_t 1) \tag{9}$$

and the budget constraint $w'1 = 1$ implies

$$1 = \; a^{-1}1'\Sigma^{-1}(\hat{\mu}-\lambda_t 1) \tag{10}$$

Solving for λ_t, we get

$$\lambda_t \;=\; \frac{a^{-1}1'\Sigma^{-1}\hat{\mu}-1}{a^{-1}1'\Sigma^{-1}1} \tag{11}$$

Another way is to fix the target expected return

$$\mu_p \;=\; w'\mu$$

and minimize the variance

$$\min w'\Sigma w,\;\; w'1 = 1,\;\; \mu_p \;=\; w'\mu\,.$$

This will give us two Lagrange multipliers

$$\Sigma w \;=\; \lambda_1 1 \;+\; \lambda_2\mu\,.$$

Thus,

$$w_t \;=\; \Sigma^{-1}(\lambda_1 1 \;+\; \lambda_2\mu)\,.$$

Defining "effective risk aversion" $a = \lambda_2^{-1}$ and $\lambda_t = -\lambda_1/\lambda_2$, we see that this portfolio is identical to that derived above. Thus, we can parametrize tangency portfolios either by a or by μ_p.

Consider now a mean-variance optimizing agent. When the agent is constrained, he is solving

$$\max w'\mu \;-\; 0.5aw'\Sigma w \;-\; \lambda(w'1-m) \tag{12}$$

and the first-order condition is

$$\mu-\lambda 1 \;=\; a\Sigma w \tag{13}$$

but λ is nonzero only when the constraint binds (remember Kuhn-Tucker conditions)! *We can then solve for λ using $w'1 = m$.*

The constraints bind only when the desired mean-variance portfolio violates it. Thus, we get

$$w_{constrained} \;=\; \begin{cases} a^{-1}\Sigma^{-1}\hat{\mu},\; 1'a^{-1}\Sigma^{-1}\hat{\mu} \le\; m \\ a^{-1}\Sigma^{-1}(\hat{\mu}-\frac{a^{-1}1'\Sigma^{-1}\hat{\mu}-m}{a^{-1}1'\Sigma^{-1}1}1), \qquad 1'a^{-1}\Sigma^{-1}\hat{\mu} >\; m \end{cases} \tag{14}$$

In particular, the claim about threshold risk aversion follows with

$$a^* \;=\; \frac{1'\Sigma^{-1}\hat{\mu}}{m} \tag{15}$$

4. Betting against Beta

Read Frazzini and Pedersen (JFE 2013): Betting against Beta (posted on Moodle).

1. Explain the main empirical experiment and the evidence obtained in FP2013.
2. Explain the main economic mechanism that FP2013 put forward to explain their empirical findings.
3. Read the "Speculative Betas" paper on Moodle and compare its findings with those of Frazzini and Pedersen (JFE 2013). What is the difference?